

## ON THE AMBIPOLAR DIFFUSION AND OTHER ASYMPTOTIC LIMITS OF WEAKLY IONIZED PLASMA FLOWS

FRANCOIS GOLSE

*Ecole Polytechnique, CMLS, 91128 Palaiseau, France*  
*golse@ann.jussieu.fr*

PIERRE-HENRI MAIRE

*CELIA, Universite Bordeaux-I, 33405 Talence, France*  
*mair@celia.u-bordeaux.fr*

REMI SENTIS\*

*C.E.A./ Bruyères, B.P. 12, 91680 Bruyères, France*  
*remi.sentis@cea.fr*

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This paper reviews several different models of weakly ionized plasma flows, where charged particles are scattered in a background flow of neutral particles. A multispecies diffusion model is derived from either a kinetic model or a multifluid model (which is the Maxwellian closure of the kinetic model). This derivation is purely formal (in the sense that only the Hilbert or Chapman–Enskog expansions are established). An even simpler model, the so-called ambipolar diffusion model is derived from the multispecies diffusion model. This latter derivation is based on energy estimates and on a form of maximum principle which is established by writing the multispecies diffusion system in terms of the so-called Slotboom variables.

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### 1. Introduction

The subject matter of this paper is the mathematical modeling of weakly ionized plasma flows. For simplicity, all plasmas considered in this paper consist of a single species of neutral particles, a single species of ions (whose ionization level is equal to 1) and electrons.

\*Corresponding author

Moreover, the plasmas are assumed to be weakly ionized, in the sense that only the binary collisions involving one ion or one electron and one neutral particle are taken into account. In other words, the flow of neutral particles is somehow prescribed (and is not much affected by that of charged particles); it drives the flow of charged particles and acts as a thermal bath. Hence the macroscopic picture for the charged particles is the following: their bulk velocity is that of the neutral particles; however, their interaction with the bath of neutral particles results in mass and possibly temperature diffusion.

There are various models for such plasma flows: there is a kinetic model, a multifluid model, a multispecies diffusion model and finally the so-called ambipolar diffusion model. This paper is aimed at explaining how these various models are related and in particular how they can be derived from one another under relevant scaling assumptions. In particular, we have tried to delineate carefully the various assumptions on orders of magnitude under which each model can be used.

The first main results are related to the “formal” derivations of the multispecies diffusion model from both the kinetic and the multifluid models. A remarkable feature of this derivation is that the kinetic and multifluid models lead to different multispecies diffusion limiting equations; indeed the multispecies diffusion equation derived from the kinetic model involves a term modeling thermal diffusion while the one derived from the multifluid model does not. This kind of multispecies diffusion models may be found for example in Refs. 8, 12 and 18.

As for the meaning of “formal”, we refer to Sec. 3 and to the precise statements of the theorems there. Suffices it to say here that these results are based on the Hilbert or Chapman–Enskog expansion methods, without any proof of convergence.

Another important point is that all the analyses are stated in the whole space  $\mathbb{R}^3$ , meaning that these approximations are expected to be valid only far from the boundary of the domain. The presence of electrized boundaries usually involves rather complicated boundary effects which one hopes to model by boundary layer techniques: see for example Ref. 5, as well as Ref. 1 in the much more complicated case of plasma erosion or electric sheaths.

The other important result is the mathematical derivation of the ambipolar diffusion equation from a multispecies diffusion model. Here, our starting point is not a first principle equation itself but the system consisting of a diffusion equation for the ionic density, a generalized Boltzmann relation for the electronic density and the Poisson equation relating the electric potential to the density of charges. This model is the formal limit of the multispecies diffusion model when the inertia of the electrons is neglected. At variance with the rest of the paper, this derivation is done in a bounded domain; however the boundary conditions are compatible with the interior approximation so that no boundary layers are required to perform the analysis.

The paper is organized as follows: in Sec. 2, we review the main different models for weakly ionized plasma flows that we want to derive from one another; we also

indicate in this section the various scaling assumptions under which each model can be derived from a more microscopic model. Section 3 is devoted to the derivation of the multispecies diffusion model from two different models: the multifluid and the kinetic model. Remarkably enough, the two limiting multispecies diffusion models are, as we said, different. Section 4 is devoted to the mathematical derivation of the ambipolar diffusion model from the multispecies diffusion model.

## 2. Various Models for the Weakly Ionized Plasmas

### 2.1. The kinetic model

The most detailed description that we shall consider here is a kinetic model derived from the general multispecies system, based on the following assumptions:

- the distribution of neutral particles is a local Maxwellian,
- collisions between charged particles are neglected (this means that the ionization level is very weak),
- collisions between neutral particles and ions or electrons are elastic.

The corresponding model is written in accordance with the general prescription for the kinetic description of mixtures. This kind of models may be found for example in Refs. 7 and 15. The only modifications come from the two first assumptions above, leading to the following system:

$$(\mathcal{K}) \quad \begin{cases} \frac{\partial f_i}{\partial t} + v \cdot \nabla_x f_i - \frac{1}{\beta_i^2} \frac{\nabla_x \psi}{m_n} \cdot \nabla_v f_i = \mathcal{Q}_{i,n}(f_i, f_n), \\ \frac{\partial f_e}{\partial t} + v \cdot \nabla_x f_e + \frac{1}{\beta_e^2} \frac{\nabla_x \psi}{m_n} \cdot \nabla_v f_e = \mathcal{Q}_{e,n}(f_e, f_n), \\ -\Delta_x \psi = \frac{e_0^2}{\varepsilon_0} (N_i - N_e). \end{cases}$$

In the system above,  $t$ ,  $x$  and  $v$  denote respectively the time, position and velocity variable;  $f_\alpha \equiv f_\alpha(t, x, v)$  is the number density of particles of the species  $\alpha$  with the subscript  $\alpha$  being one of  $i$  (for ions),  $e$  (for electrons) and  $n$  (for neutral). The mass of particles of type  $\alpha$  is denoted by  $m_\alpha$ , while  $e_0$  denotes the (absolute value of the) charge of the electron and  $\varepsilon_0$  is the electric permittivity of the vacuum. For notational simplicity, we have denoted the mass ratios by  $\beta_\alpha^2 = m_\alpha/m_n$  and the electric potential (up to the multiplicative constant  $e_0$ ) by  $\psi$ . Finally  $N_\alpha \equiv N_\alpha(t, x)$  is the macroscopic number density of the species  $\alpha$ , namely

$$N_\alpha(t, x) = \int_{\mathbb{R}^3} f_\alpha(t, x, v) dv.$$

The terms  $\mathcal{Q}_{i,n}(f_i, f_n)$  and  $\mathcal{Q}_{e,n}(f_e, f_n)$  are the Boltzmann collision integrals modeling respectively the instantaneous variation of the ionic and electronic

densities due to collisions with the neutral particles. Their expression is as follows:

$$\begin{aligned} \mathcal{Q}_{\alpha,n}(f_\alpha, f_n)(t, x, v) &= \int_{\mathbb{R}^3} dv_n \int_{S^2} (f_\alpha(t, x, v'_\alpha) f_n(t, x, v'_n) - f_\alpha(t, x, v) f_n(t, x, v_n)) r_{\alpha,n}(v - v_n, \omega) d\omega, \end{aligned}$$

where  $v'_\alpha$  and  $v'_n$  are the post-collision velocities corresponding to pre-collision velocities  $v$  and  $v_n$  of the particles of type  $\alpha$  and the neutral particles respectively. These collisions preserve the total momentum and kinetic energy of any colliding pair of particles

$$m_\alpha v + m_n v_n = m_\alpha v'_\alpha + m_n v'_n, \quad m_\alpha v^2 + m_n v_n^2 = m_\alpha |v'_\alpha|^2 + m_n |v'_n|^2.$$

It is well-known that, given  $v$  and  $v_n$ , the set of solutions  $v'_\alpha$  and  $v'_n$  of the system above is parametrized by the unit vector  $\omega$ . The positive function  $r_{\alpha,n}$  denotes the product of the cross-section for the particles of type  $\alpha$  and the neutral particle collisions by the relative velocity of the colliding pair.

As mentioned in the Introduction, one should think of the neutral particle distribution as a given function. In the sequel, this distribution function is assumed to be a Maxwellian function parametrized by  $(N_n, U_n, T_n)$  that is to say

$$f_n(v) = N_n \mathcal{M}_n(v), \quad \text{with } \mathcal{M}_n(v) = \frac{1}{(2\pi T_n)^{3/2}} e^{-\frac{|v-U_n|^2}{2T_n}},$$

where the density  $N_n$ , the velocity  $U_n$  and the temperature  $T_n$  are given smooth functions of  $t$  and  $x$ , as a matter of fact they are assumed to be solution of the Euler system for compressible fluids, see (3.1) below. Notice that  $T_n$  denotes the temperature normalized by the mass  $m_n$  (i.e.  $T_n = \mathcal{T}_n/m_n$  where  $\mathcal{T}_n$  is the physical temperature).

### 2.2. *The multifluid model*

Let  $l_M$  be a macroscopic length scale of the neutral flow (for example, the size of a body immersed in the flow, or of the domain where the plasma is confined. . .). In this paper, we assume that

— the mean free path of charged particles is small if compared to  $l_M$ .

More precisely, let  $\mu_i = \mu_i(T_n)$  be the mean collision frequency between the ions and the neutral particles (which will be defined in Sec. 3.1); if  $\mu_C$  and  $\mathcal{T}_C$  denote characteristic values of  $\mu_i$  and of the physical temperature  $\mathcal{T}_n$ , the characteristic mean free path is  $l_C = (\frac{\mathcal{T}_C}{m_n})^{1/2} \frac{1}{\mu_C}$  and it is convenient to introduce a first small parameter

$$\eta = \frac{l_C}{l_M} \ll 1. \tag{2.1}$$

Then, in order to avoid the prohibitive numerical cost of multispecies kinetic models, one often replaces the kinetic model above by a hydrodynamic closure which is of the form of the classical Euler–Poisson system but with nonconservative terms describing the relaxation of momentum of the ions and electrons due to the collisions with neutral particles.

For this model, we also assume that:

— the ion and electron temperatures are somehow given and are equal to the temperature of the neutral fluid.

Notice that this last assumption may be invalid, even when the collision frequency of ions against neutral particles is very small if compared to the ratio of  $l_M$  over the characteristic value of the their velocity. In such a case, the ions and the neutral particles are at the same temperature, while the electronic temperature is governed by the classical energy equation which involves the coupling to the temperature of the other species; then the modeling is more complicated but the general ideas are the same as the ones presented in this paper.

This hydrodynamic closure reads as the following multifluid model (see for example Ref. 6)

$$(\mathcal{MF}) \quad \left\{ \begin{array}{l} \frac{\partial}{\partial t} N_i + \nabla_x \cdot (N_i U_i) = 0, \\ \beta_i^2 \frac{\partial}{\partial t} (N_i U_i) + \beta_i^2 \nabla_x \cdot (N_i U_i \otimes U_i) + \nabla_x (N_i T_n) + N_i \frac{\nabla_x \psi}{m_n} \\ \qquad \qquad \qquad = -\beta_i \mu_i N_i (U_i - U_n), \\ \frac{\partial}{\partial t} N_e + \nabla_x \cdot (N_e U_e) = 0, \\ \beta_e^2 \frac{\partial}{\partial t} (N_e U_e) + \beta_e^2 \nabla_x \cdot (N_e U_e \otimes U_e) + \nabla_x (N_e T_n) - N_e \frac{\nabla_x \psi}{m_n} \\ \qquad \qquad \qquad = -\beta_e \mu_e N_e (U_e - U_n), \\ -\frac{\epsilon_0}{e_0^2} \Delta_x \psi = N_i - N_e. \end{array} \right.$$

In the above model,  $N_\alpha \equiv N_\alpha(t, x)$  is the macroscopic number density of the species  $\alpha$ ,  $U_\alpha \equiv U_\alpha(t, x)$  its velocity field.

**2.3. The multispecies diffusion model**

With the previous assumptions, the multispecies diffusion model is a good approximation of the kinetic one. It can be found from the general prescription in Ref. 15. For our situation, with the same notations as above, it may be recast in the following

form, where  $P_n = N_n T_n$  denotes the neutral pressure:

$$(\mathcal{D}) \quad \left\{ \begin{aligned} & \frac{\partial N_i}{\partial t} + \nabla_x \cdot (N_i U_n) \\ & = \frac{1}{\beta_i} \nabla_x \cdot \left[ N_i D_i^T \frac{\nabla_x T_n}{T_n} + N_i D_i \left( \frac{\nabla_x \psi}{m_n T_n} + \frac{\nabla_x (N_i T_n)}{N_i T_n} - \beta_i^2 \frac{\nabla_x P_n}{P_n} \right) \right], \\ & \frac{\partial N_e}{\partial t} + \nabla_x \cdot (N_e U_n) \\ & = \frac{1}{\beta_e} \nabla_x \cdot \left[ N_e D_e^T \frac{\nabla_x T_n}{T_n} + N_e D_e \left( -\frac{\nabla_x \psi}{m_n T_n} + \frac{\nabla_x (N_e T_n)}{N_e T_n} - \beta_e^2 \frac{\nabla_x P_n}{P_n} \right) \right], \\ & -\frac{\varepsilon_0}{e_0^2} \Delta_x \psi = N_i - N_e. \end{aligned} \right.$$

This model can be derived rigorously from the kinetic model ( $\mathcal{K}$ ) if we let  $\eta$  go to 0; see Sec. 3.1 below where the expression of diffusion coefficients  $D_e, D_i, D_e^T, D_i^T$  are given. It is also formally derived in Sec. 3.2 below from the multifluid model if we let  $\eta$  go to 0; in which case the thermal diffusion terms vanish — in other words,  $D_e^T = D_i^T = 0$ .

**Remark 2.1.** This model (or a more general variant corresponding to several species of both neutral and charged particles) is widely used in the aerodynamics literature (see for example Refs. 8 and 12). Often the Poisson equation is replaced by a relation expressing the nullity of the electric diffusion current (i.e. the difference of the two bracketed quantities on the right-hand sides of the diffusion equations in ( $\mathcal{D}$ )). Then this model can be compared with the ambipolar diffusion one (see below).

In order to derive the so-called *Drift-Diffusion* Model with generalized Maxwell–Boltzmann law, we now set  $D_e^T = 0$  and make another approximation:

— the mass of the electron is small compared to the mass of the neutral particles and the relaxation effect negligible compared to either the electric or hydrodynamic force: that is,

$$\beta_e \ll 1, \quad \beta_e \mu_e(T_n) \ll 1.$$

Thus all that remains from the electron momentum conservation law is

$$\nabla_x (N_e T_n) - N_e \frac{\nabla_x \psi}{m_n} = 0. \tag{2.2}$$

This equation will be referred to as the generalized Boltzmann relation, for, if  $T_n$  is constant, it is equivalent to

$$N_e = C e^{\psi/(m_n T_n)} \tag{2.3}$$

which is the usual Maxwell–Boltzmann relation for the electrons. Then one can replace the model  $(\mathcal{D})$  by

$$(\mathcal{DB}) \quad \left\{ \begin{array}{l} \frac{\partial N_i}{\partial t} + \nabla_x \cdot (N_i U_n) \\ \quad = \frac{1}{\beta_i} \nabla_x \cdot \left[ N_i D_i^T \frac{\nabla_x T_n}{T_n} + N_i D_i \left( \frac{\nabla_x \psi}{m_n T_n} + \frac{\nabla_x (N_i T_n)}{N_i T_n} - \beta_i^2 \frac{\nabla_x P_n}{P_n} \right) \right], \\ \nabla_x (N_e T_n) - N_e \frac{\nabla_x \psi}{m_n} = 0, \\ -\frac{\epsilon_0}{c_0^2} \Delta_x \psi = N_i - N_e. \end{array} \right.$$

**2.4. Quasi-neutrality: The ambipolar diffusion model**

Denote by  $N_C$  a characteristic value of  $N_e$ . Let us recall that in a plasma, one can define a Debye length  $l_D$  by:

$$l_D^2 = \frac{\epsilon_0 T_C}{e_0^2 N_C}.$$

We simplify further the multispecies diffusion model by making the classical assumption that the Debye length is small compared to the macroscopic length; then we introduce the small parameter:

$$\epsilon = l_D / l_M \ll 1.$$

On the Poisson equation

$$-\frac{N_C l_M^2}{T_C} \epsilon^2 \Delta \psi = N_i - N_e$$

it becomes manifest that, in the limit as  $\epsilon \rightarrow 0$ ,  $|N_i - N_e| \ll N_i + N_e$ , at least in some weak sense. Hence, one can define an approximate common value for the ionic electronic densities by

$$N_* = \frac{1}{2}(N_i + N_e) \simeq N_i \simeq N_e.$$

Next one can eliminate the electric force term in the ion momentum equation by combining the two first equations of system  $(\mathcal{DB})$ . (Notice that the electric field is eliminated at the expense of changing the pressure law from  $P_i = N_i T_n$  to  $P_* = 2N_* T_n$ . This feature is unique to all quasi-neutral models.) After eliminating this electric force in the diffusion flux of the multispecies diffusion model, one arrives at the single equation governing the dynamics of  $N_*$ :

$$(\mathcal{APD}) \quad \frac{\partial N_*}{\partial t} + \nabla_x \cdot (N_* U_n) - \frac{1}{\beta_i} \nabla_x \cdot (N_* 2D_i \nabla_x \log(N_* T_n)) = 0.$$

That is to say, one eliminates the electric force at the expense of doubling the ionic diffusion coefficient: introducing this factor 2 is the reason why this diffusion

equation is called ‘‘ambipolar’’. This kind of models may be found in the classical textbook of Delcroix (cf. Ref. 10).

### 3. Derivation of the Multispecies Diffusion Model from the Kinetic and the Multifluid Models

We now explain how to derive the multispecies diffusion model ( $\mathcal{D}$ ) first from the kinetic model ( $\mathcal{K}$ ), then from the multifluid model ( $\mathcal{MF}$ ). In fact these derivations are extremely similar in spirit. We shall, however, treat both for they do not lead to exactly the same diffusion equation. We have to precise the evolution of the neutral flow; so we assume from now on that  $N_n, U_n, T_n$  satisfy the Euler system

$$\begin{aligned} \frac{\partial}{\partial t} N_n + \nabla_x \cdot (N_n U_n) &= 0, \\ \frac{\partial}{\partial t} N_n U_n + \nabla_x \cdot (N_n U_n \otimes U_n) + \nabla_x (N_n T_n) &= 0, \\ \frac{\partial}{\partial t} N_n T_n + \nabla_x \cdot (N_n T_n U_n) + \frac{2}{3} N_n T_n \nabla_x \cdot U_n &= 0. \end{aligned} \tag{3.1}$$

#### 3.1. Derivation from the kinetic model

From the kinetic model ( $\mathcal{K}$ ), we make here a Hilbert expansion in the small parameter  $\eta$ .

In this section, all the velocities are normalized by the mass ratio, that is say

$$\bar{v}_i = \beta_i v_i, \quad \bar{v}_e = \beta_e v_e$$

and

$$\bar{f}_i(\bar{v}_i) = f_i(v_i), \quad \bar{f}_e(\bar{v}_e) = f_e(v_e).$$

The normalized Maxwellian function corresponding to the species  $\alpha$  is given by the formula:

$$M_\alpha(\bar{v}) = \frac{1}{(2\pi T_n)^{3/2}} \exp\left[-\frac{|\bar{v} - \beta_\alpha U_n|^2}{2T_n}\right].$$

On the other hand, the mean collision frequency between the ions and the neutral particles  $\mu_i = \mu_i(x, \bar{v}_i)$  is defined by

$$\mu_i(x, \bar{v}_i) = N_n \iint \mathcal{M}_n(v_n) r(\bar{v}_i \beta_i^{-1}, \omega) d\omega dv_n,$$

where  $\mathcal{M}_n$  is given above. Here, if  $\mu_C$  is a characteristic value of  $\mu_i$ , we assume that

$$\eta = \frac{(\mathcal{I}_C/m_n)^{1/2}}{l_M \mu_C} = \frac{l_C}{l_M}$$

is a small parameter. We now perform the following change of variables

$$q_{i,n}(w\beta_i, \omega) = \eta r_{i,n}(w, \omega), \quad q_{e,n}(w\beta_e, \omega) = \eta r_{e,n}(w, \omega).$$

From now on, we write  $f_\alpha, v_\alpha$  instead of  $\bar{f}_\alpha, \bar{v}_\alpha$ . The kinetic equation satisfied by the distribution function  $f_\alpha$  reads as:

$$\frac{\partial f_\alpha}{\partial t} + \frac{1}{\beta_\alpha} \left( v \cdot \nabla_x f_\alpha - \xi_\alpha \frac{\nabla_x \psi}{m_n} \cdot \nabla_v f_\alpha \right) = \frac{1}{\eta \beta_\alpha} \mathbf{Q}_\alpha(f_\alpha), \tag{3.2}$$

where we have set  $\xi_e = -1$  and  $\xi_i = +1$ ; and the collision operators are

$$\mathbf{Q}_\alpha(f_\alpha)(v) = N_n \int_{\mathbb{R}^3} dv_n \int_{S^2} (f_\alpha(v') \mathcal{M}_n(v'_n) - f_\alpha(v) \mathcal{M}_n(v_n)) q_{\alpha,n}(\beta_\alpha v_n - v, \omega) d\omega,$$

with

$$v' = v - \frac{2}{1 + \beta_\alpha^2} ((\beta_\alpha v_n - v) \cdot \omega) \omega, \quad v'_n = v_n - \frac{2\beta_\alpha}{1 + \beta_\alpha^2} ((\beta_\alpha v_n - v) \cdot \omega) \omega. \tag{3.3}$$

Moreover, the electric potential  $\psi$  is given by the Poisson equation

$$-\frac{\epsilon_0}{e_0^2} \Delta_x \psi = \int f_i(v) dv - \int f_e(v) dv.$$

The goal of this section is to construct formal approximate solutions to (3.2). Formal solutions of an equation depending upon the small parameter  $\eta$  are usually defined as formal series in the parameter  $\eta$  with  $C^\infty$  coefficients satisfying the equation. Of course the series will be truncated, in the present case at order 2.

Denote now  $f_\alpha^\eta$  the solution (3.2), it is sought in the form

$$f_\alpha^\eta = f_\alpha^0(1 + \eta f_\alpha^1 + \eta^2 f_\alpha^2 + \dots). \tag{3.4}$$

On principle, one should also expand the electric field  $-\nabla_x \psi$  in terms of  $\eta$ . However, this is useless in the case considered here: indeed, the electric field in the multifluid model ( $\mathcal{MF}$ ) appears along with the diffusion terms and is of the same order of magnitude, meaning that it is small compared to the inertial term. One can check this particular point in the statement of Theorem 3.1 below: the electric field appear at the order  $O(\eta)$  in the limiting diffusion model. Hence, for simplicity it suffices to keep the leading order term in the electric field; in other words, expanding the electric field is unnecessary under the scaling assumptions leading to the multi-species diffusion model ( $\mathcal{MD}$ ).

For the sake of compactness, we use the notations

$$G = \frac{\xi_\alpha}{m_n} \nabla_x \psi \quad \text{and} \quad \frac{D}{Dt} = \beta_\alpha \frac{\partial}{\partial t} + v \cdot \nabla_x - G \cdot \nabla_v.$$

It will be especially convenient to consider the collision operator twisted by the Maxwellian, instead of the original form. In other words, observe that  $\beta_\alpha v + v_n = \beta_\alpha v' + v'_n$  and  $v_\alpha^2 + v_n^2 = v_\alpha'^2 + v_n'^2$ ; therefore one has

$$\mathcal{M}_n(v'_n) M_\alpha(v') = \mathcal{M}_n(v_n) M_\alpha(v),$$

which suggests to recast the collision operator of the form

$$\mathbf{Q}_\alpha(f_\alpha)(v) = N_n M_\alpha \int_{\mathbb{R}^3} \int_{\omega \in S^2_+} \mathcal{M}_n(v_n) (f_\alpha(v') M_\alpha(v')^{-1} - f_\alpha(v) M_\alpha(v)^{-1}) q_{\alpha n} d\omega dv_n.$$

We consider instead the twisted operator  $\Gamma$  defined for any function  $g = g(v)$  by:

$$\Gamma(g)(v) = N_n^{-1} M_\alpha^{-1} \mathbf{Q}_\alpha(M_\alpha g)(v) = \iint \mathcal{M}_n(v_n) (g(v') - g(v)) q_{\alpha n} d\omega dv_n. \tag{3.5}$$

With these notations, the linear kinetic equation (3.2) reads

$$\begin{aligned} \frac{D}{Dt} (f_\alpha^0 (1 + \eta f_\alpha^1 + \eta^2 f_\alpha^2 + \dots)) \\ = \eta^{-1} N_n M_\alpha \Gamma (M_\alpha^{-1} f_\alpha^0 (1 + \eta f_\alpha^1 + \eta^2 f_\alpha^2 + \dots)). \end{aligned} \tag{3.6}$$

The Hilbert expansion method consists in balancing order by order in  $\eta$  each side of (3.6). It leads to

- Order  $\eta^{-1}$ :

$$\Gamma(M_\alpha^{-1} f_\alpha^0) = 0. \tag{3.7}$$

- Order  $\eta^0$ :

$$N_n^{-1} \frac{Df_\alpha^0}{Dt} = M_\alpha \Gamma(M_\alpha^{-1} f_\alpha^0 f_\alpha^1). \tag{3.8}$$

- Order  $\eta^1$ :

$$N_n^{-1} \frac{Df_\alpha^0 f_\alpha^1}{Dt} = M_\alpha \Gamma(M_\alpha^{-1} f_\alpha^0 f_\alpha^2). \tag{3.9}$$

Solving this string of equations depends upon applying the Fredholm alternative to the integral operator  $\Gamma$ . Thus we have to state the properties of the collision operator.

### 3.1.1. *Properties of the collision operator*

Consider the Hilbert space  $\mathcal{L}_\alpha = \{g \mid \int_{\mathbb{R}^3} g^2(v) M_\alpha(v) dv < +\infty\}$ , with scalar product  $\langle \cdot \mid \cdot \rangle$  defined by

$$\langle g \mid h \rangle_\alpha = \int_{v \in \mathbb{R}^3} g(v) h(v) M_\alpha(v) dv$$

(in the sequel, we do not write the subscript  $\alpha$ ). For proving the following proposition, we assume that there exist  $0 < C < C'$  and  $\gamma \in [0, 1]$ , such that

$$C(1 + |v - U_n|)^\gamma \leq \iint \mathcal{M}_n(v_n) q_{\alpha n} d\omega dv_n \leq C'(1 + |v - U_n|)^\gamma, \quad \forall v \in \mathbb{R}^3. \tag{3.10}$$

**Proposition 3.1.** *The operator  $\Gamma$  is a nonpositive Fredholm operator on  $\mathcal{L}_\alpha$  with domain*

$$\mathcal{D}(\Gamma) = \{f \in \mathcal{L}_\alpha \text{ such that } |v|^\gamma f \in \mathcal{L}_\alpha\};$$

it has the following properties

- (a)  $\forall (g, h) \in \mathcal{D}(\Gamma), \langle \Gamma(g)|h \rangle = \langle g|\Gamma(h) \rangle,$
- (b)  $\forall g \in \mathcal{D}(\Gamma), \langle \Gamma(g)|g \rangle \leq 0,$
- (c)  $\Gamma(g) = 0 \iff g(v) \equiv C,$  where  $C$  is a const.,
- (d)  $\forall h \in \mathcal{L}_\alpha^2, \exists g \in \mathcal{L}_\alpha$  such that  $\Gamma(g) = h \iff \langle h|1 \rangle = 0.$

**Sketch of the Proof.** For any function  $h$ , we write  $h'$  for  $h(v')$  where  $v'$  is given by (3.3). According to the classical symmetry relation  $(v, v_n) \rightarrow (v', v'_n)$ , we see that for all  $h$ ,

$$\int \mathbf{Q}(f)(v)h(v)dv = -\frac{1}{2}N_n \iiint M_\alpha(v)\mathcal{M}_n(v_n) (f'M_\alpha'^{-1} - fM_\alpha^{-1}) (h' - h)q_{\alpha n}d\omega dv_n dv.$$

Hence, for all  $h$  and  $g \in \mathcal{D}(\Gamma)$ ,

$$\langle h|\Gamma(g) \rangle = -\frac{1}{2} \iiint M_\alpha(v)\mathcal{M}_n(v_n)(g' - g)(h'_\alpha - h_\alpha)q_{\alpha n}d\omega dv_n dv \tag{3.11}$$

which establishes properties (a) and (b). Applying (3.11) with  $h = g$  implies property (c). On the other hand, we can decompose  $\Gamma$  in the form<sup>14</sup>

$$(\Gamma g)(v) = -g(v) \iint \mathcal{M}_n(v_n)q_{\alpha n}d\omega dv_n + (Kg)(v), \quad \text{with } K \text{ compact,} \tag{3.12}$$

then  $\Gamma$  is Fredholm and (d) follows. □

3.1.2. *The asymptotic expansion with respect to  $\eta$*

*Terms at order  $\eta^{-1}$ .* The relation (3.7) together with Proposition 3.1(c) show that

$$f_\alpha^0(t, x, v) = N_\alpha^0(t, x)M_\alpha(v). \tag{3.13}$$

In other words, the equation at order  $\eta^{-1}$  gives the  $v$  dependence of  $f_\alpha^0$ ; but  $f_\alpha^0$  is modulated in the variables  $t$  and  $x$  and the equation governing this modulation is determined by the next order in the expansion.

*Terms at order  $\eta^0$ .* From (3.8) and (3.13), we get

$$\Gamma f_\alpha^1 = \frac{1}{N_n M_\alpha N_\alpha^0} \frac{D}{Dt} (M_\alpha N_\alpha^0). \tag{3.14}$$

The compatibility condition (d) in Proposition 3.1 leads to a PDE for  $N_\alpha^0$ :

$$\left\langle \frac{D}{Dt} (M_\alpha N_\alpha^0) | M_\alpha^{-1} \right\rangle = \left\langle \beta_\alpha \frac{\partial}{\partial t} (M_\alpha N_\alpha^0) + v \cdot \nabla_x (M_\alpha N_\alpha^0) - G \cdot \nabla_v (M_\alpha N_\alpha^0) | M_\alpha^{-1} \right\rangle = 0.$$

Since  $\langle v|M_\alpha^{-1} \rangle = \beta_\alpha U_n$  and  $\langle \nabla_v g|M_\alpha^{-1} \rangle = 0$  for any  $g$  converging to 0 at infinity, one gets

$$\frac{\partial N_\alpha^0}{\partial t} + \nabla_x \cdot (N_\alpha^0 U_n) = 0. \tag{3.15}$$

This compatibility condition being satisfied, we can solve for  $f_\alpha^1$ . Before embarking on the computation of the right-hand side of (3.14), it is convenient to recall some standard notations.

The identity matrix is denoted by  $\mathbf{I}$ ; for all  $3 \times 3$  matrices  $S$  and  $S'$ , let us denote  $S : S' = \sum_{1 \leq i, j \leq 3} S_{ij} S'_{ij}$ ; in addition, the following vectors and matrices are of constant use in the Hilbert expansion:

$$W = \frac{v - \beta_\alpha U_n}{\sqrt{T_n}}, \quad A(W) = (|W|^2 - 5)W, \quad B(W) = W \otimes W - \frac{\mathbf{I}}{3}|W|^2. \tag{3.16}$$

Finally, let us state  $X_\alpha = N_\alpha^0/N_n$ ; then, with these notations at hand, we get

**Lemma 3.1.** *If  $N_\alpha^0$  satisfies relation (3.15) while  $N_n, U_n$  and  $T_n$  satisfy the Euler system (3.1), then one gets*

$$\begin{aligned} \frac{1}{N_\alpha^0 M_\alpha} \frac{D}{Dt} (N_\alpha^0 M_\alpha) &= \beta_\alpha B(W) : \nabla_x U_n + A(W) \cdot \nabla_x \sqrt{T_n} \\ &+ \frac{W}{\sqrt{T_n}} \cdot \left[ G + T_n \left( \nabla_x \log X_\alpha + (1 - \beta_\alpha^2) \frac{\nabla_x P_n}{P_n} \right) \right]. \end{aligned}$$

**Proof.** A classical computation gives the logarithmic derivative of the Maxwellian distribution:

$$\begin{aligned} \left( \beta_\alpha \frac{\partial}{\partial t} + v \cdot \nabla_x \right) \log M_\alpha &= \beta_\alpha B(W) : \nabla_x U_n + A(W) \cdot \nabla_x \sqrt{T_n} + 2W \cdot \nabla_x \sqrt{T_n} \\ &+ \beta_\alpha \nabla_x \cdot U_n - \frac{\beta_\alpha^2}{N_n \sqrt{T_n}} W \cdot \nabla_x P_n. \end{aligned} \tag{3.17}$$

Then, the relation (3.15) leads to

$$\frac{1}{N_\alpha^0} \beta_\alpha \frac{\partial N_\alpha^0}{\partial t} = -\beta_\alpha U_n \cdot \nabla_x \log N_\alpha^0 - \beta_\alpha \nabla_x \cdot U_n. \tag{3.18}$$

Combining (3.17) with (3.18) leads to the announced expression, after noticing that  $\nabla_v \log M_\alpha = T_n^{-1/2} W$ . □

The next lemma is classical; it reflects the invariance of the collision operator  $\Gamma$  under the group  $O_3(\mathbb{R})$  of the rotations of  $\mathbb{R}^3$ .

**Lemma 3.2.** *Let  $\Gamma^{-1} : (\text{Ker } \Gamma)^\perp \rightarrow (\text{Ker } \Gamma)^\perp$  be the pseudo-inverse of  $\Gamma$ . There exist three real-valued functions  $a_\alpha, b_\alpha$  and  $c_\alpha$  defined on  $\mathbb{R}^+$  such that, componentwise:*

$$\begin{aligned} \Gamma^{-1}(A(W)) &= -a_\alpha(|W|)A(W), \\ \Gamma^{-1}(B(W)) &= -b_\alpha(|W|)B(W), \\ \Gamma^{-1}(W) &= -c_\alpha(|W|)W. \end{aligned}$$

**Proof.** See Ref. 13 for a complete proof of this fact; notice that it is especially important to consider the relations above as involving vectors or tensors, and to let the group  $O_3(\mathbb{R})$  operate on them. The key idea is the observation that, for any  $R \in O_3(\mathbb{R})$  we have

$$A(RW) = RA(W), \quad B(RW) = RB(W)R^T. \quad \square$$

Consider now the relation (3.14), assuming that the compatibility condition (3.15) is satisfied. By the two previous lemmas together with Proposition 3.1(c), one gets

$$f_\alpha^1(t, x, v) = N_\alpha^1 - \frac{1}{N_n} \beta_\alpha b_\alpha(|W|)B(W) : \nabla_x U_n - \frac{1}{N_n} a_\alpha(|W|)A(W) \cdot \nabla_x \sqrt{T_n} - \frac{1}{N_n \sqrt{T_n}} c_\alpha(|W|)W \cdot \left[ G + T_n \left( \nabla_x \log X_\alpha + (1 - \beta_\alpha^2) \frac{\nabla_x P_n}{P_n} \right) \right].$$

The equation governing the evolution of  $N_\alpha^1$  is found by looking at the compatibility condition at order  $\eta^1$ . Notice that a trivial parity argument shows that for all  $i, j, k = 1, 2, 3$

$$\langle b_\alpha(|W|)B_{ij}(W)|W_k \rangle = \langle a_\alpha(|W|)A_i(W)|1 \rangle = \langle c_\alpha(|W|)W_i|1 \rangle = 0$$

while, for all  $i \neq j \in \{1, 2, 3\}$

$$\langle a_\alpha(|W|)A_i(W)|W_j \rangle = \langle c_\alpha(|W|)W_i|W_j \rangle = 0.$$

In addition, the  $O_3(\mathbb{R})$ -invariance gives, for all  $i, j = 1, 2, 3$

$$\langle b_\alpha(|W|)B_{ij}(W)|1 \rangle = 0.$$

*Terms at order  $\eta^1$ .* According to the previous relations, the compatibility condition (d) in Proposition 3.1 applied to (3.9) leads to

$$\begin{aligned} 0 &= \left\langle M_\alpha^{-1} \frac{D}{Dt} (N_\alpha^0 M_\alpha f_\alpha^1) | 1 \right\rangle \\ &= \beta_\alpha \frac{\partial N_\alpha^0 N_\alpha^1}{\partial t} + \beta_\alpha \nabla_x \cdot (N_\alpha^0 N_\alpha^1 U_n) \\ &\quad - \nabla_x \cdot \left[ \frac{N_\alpha^0}{N_n} \sqrt{T_n} \langle a_\alpha(|W|)A(|W|) \otimes W | 1 \rangle \nabla_x \sqrt{T_n} \right] \\ &\quad - \nabla_x \cdot \left[ \frac{N_\alpha^0}{N_n} \langle c_\alpha(|W|)W \otimes W | 1 \rangle [G + T_n (\nabla_x \log X_\alpha + (1 - \beta_\alpha^2) \nabla_x \log P_n)] \right]. \end{aligned}$$

Using again the previous relations, we see that the diffusion tensors are diagonal and isotropic; so we can introduce the mass diffusion and thermal diffusion coefficients

$$\tilde{D}_\alpha = \frac{T_n}{3N_n} \langle c_\alpha(|W|)|W|^2|1 \rangle, \quad \tilde{D}_\alpha^T = \frac{T_n}{6N_n} \langle a_\alpha(|W|)A(W) \cdot W | 1 \rangle. \quad (3.19)$$

We refer to the Appendix for more details on  $\tilde{D}_\alpha$  and  $\tilde{D}_\alpha^T$ . In particular,  $\tilde{D}_\alpha > 0$ , while there is no proof regarding the sign of  $\tilde{D}_\alpha^T$ . Then we get

$$\begin{aligned} &\beta_\alpha \frac{\partial N_\alpha^0 N_\alpha^1}{\partial t} + \beta_\alpha \nabla_x \cdot (N_\alpha^0 N_\alpha^1 U_n) \\ &= \nabla_x \cdot \left[ \tilde{D}_\alpha^T \frac{N_\alpha^0}{T_n} \nabla_x T_n + \tilde{D}_\alpha N_\alpha^0 \left( \frac{G}{T_n} + \nabla_x \log X_\alpha + (1 - \beta_\alpha^2) \nabla_x \log P_n \right) \right]. \end{aligned} \tag{3.20}$$

Once  $N_\alpha^1$  satisfies (3.20), one can solve for  $f_\alpha^2$ . We shall not bother to give the explicit form of  $f_\alpha^2$ , for the compatibility conditions above are sufficient to give the multispecies diffusion model. Notice that  $f_\alpha^2 \in (\text{Ker } \Gamma)^\perp$ , that is to say

$$\int f_\alpha^0(v) f_\alpha^2(v) dv = 0.$$

3.1.3. *Statement of the theorem*

The idea is then to consider  $N_\alpha^\eta = N_\alpha^0(1 + \eta N_\alpha^1)$  as the new unknown; adding  $\eta$  times (3.20) to (3.15) leads to

$$\begin{aligned} \beta_\alpha \frac{\partial N_\alpha^\eta}{\partial t} + \beta_\alpha \nabla_x \cdot (N_\alpha^\eta U_n) &= \eta \nabla_x \cdot \left( \tilde{D}_\alpha^T \frac{N_\alpha^\eta}{T_n} \nabla_x T_n \right) + \eta \nabla_x \cdot \left[ \tilde{D}_\alpha N_\alpha^\eta \left( \frac{1}{T_n} G \right. \right. \\ &\quad \left. \left. + \nabla_x \log \frac{N_\alpha^\eta}{N_n} + (1 - \beta_\alpha^2) \nabla_x \log P_n \right) \right] + O(\eta^2), \end{aligned}$$

where the  $O(\eta^2)$  is to be understood in the formal sense. That is to say  $N_e^\eta$  and  $N_i^\eta$  are approximated solutions of system (D).

In any case, this construction suggests the following formal theorem.

**Theorem 3.1.** *Assume that the differential cross-section satisfies (3.10). Consider a family  $(N_\alpha^\eta)_{\eta>0}$  of smooth solutions of*

$$\begin{aligned} &\beta_\alpha \frac{\partial N_\alpha^\eta}{\partial t} + \beta_\alpha \nabla_x \cdot (N_\alpha^\eta U_n) \\ &= \eta \nabla_x \cdot \left[ \tilde{D}_\alpha^T \frac{N_\alpha^\eta}{T_n} \nabla_x T_n + \tilde{D}_\alpha N_\alpha^\eta \left( \frac{1}{T_n} G + \nabla_x \log \frac{N_\alpha^\eta}{N_n} + (1 - \beta_\alpha^2) \nabla_x \log P_n \right) \right]. \end{aligned} \tag{3.21}$$

*If we define the family*

$$\begin{aligned} F_\alpha^\eta(t, x, v) &= N_\alpha^\eta M_\alpha(v) - \eta \frac{M_\alpha(v)}{N_n} \\ &\quad \times (\beta_\alpha b_\alpha(|W|) B(W) : \nabla_x U_n - a_\alpha(|W|) A(W) \cdot \nabla_x \sqrt{T_n}) \\ &\quad - \eta \frac{M_\alpha(v)}{N_n \sqrt{T_n}} c_\alpha(|W|) W \cdot \left( G + T_n \left( \nabla_x \log \frac{N_\alpha^\eta}{N_n} + (1 - \beta_\alpha^2) \nabla_x \log P_n \right) \right), \end{aligned}$$

then there exists a family of functions  $\phi_\alpha^\eta \equiv \phi_\alpha^\eta(t, x, v)$  such that  $\int \phi_\alpha^\eta(t, x, v) dv = 0$  and  $F_\alpha^\eta + \eta^2 \phi_\alpha^\eta$  is an approximate solution of order 2 to (3.2), that is to say

$$\left( \frac{D}{Dt} - \frac{1}{\eta} \mathbf{Q}_{\alpha,n} \right) (F_\alpha^\eta + \eta^2 \phi_\alpha^\eta) = O(\eta^2)$$

Notice that  $\int (F_\alpha^\eta - N_\alpha^\eta M_\alpha)(v) dv = 0$ , so at a macroscopic level  $N_\alpha^\eta$  is a good approximation of the integral of the solution  $f_\alpha$  of (3.2). If we return to the physical variables, Eq. (3.21) may read as the diffusion model ( $\mathcal{D}$ ) stated in Sec. 2 with

$$D_\alpha = \eta \tilde{D}_\alpha, \quad D_\alpha^T = \eta \tilde{D}_\alpha^T.$$

### 3.2. Derivation from the multifluid model

In this subsection, we shall present a formal derivation of the multispecies diffusion model ( $\mathcal{D}$ ) from the multifluid model ( $\mathcal{MF}$ ). The multifluid model is an example of a system of conservation laws with an external force (the electric force) and relaxation term (due to the slowing of charged particles by collisions with the neutral background particles). For systems of conservation laws in the strong relaxation limit, formal expansions in the style of the Chapman–Enskog expansion have been proposed by different authors: see in particular Chen–Levermore–Liu<sup>9</sup> and the references therein. Just as the Chapman–Enskog expansion allows to derive (at least at the formal level) the compressible and incompressible Navier–Stokes equations from the Boltzmann equation,— see for example Bardos–Golse–Levermore,<sup>3</sup> Bardos–Ukai,<sup>4</sup> and DeMasi–Esposito–Lebowitz<sup>11</sup> for an account of the formal derivations as well as for the mathematical progress made recently in this program — the analogous expansions allow to derive (nonlinear) drift-diffusion systems from the original systems of conservation laws in the strong relaxation limit. We shall follow in this section the method of Chen–Levermore–Liu,<sup>9</sup> the only difference being the (electric) force term.

Using the same small parameter  $\eta = l_C/l_M$  as above, the multifluid model ( $\mathcal{MF}$ ) reads as

$$\begin{aligned} \frac{\partial}{\partial t} N_i + \nabla_x \cdot (N_i U_i) &= 0, \\ \beta_i^2 \frac{\partial}{\partial t} (N_i U_i) + \beta_i^2 \nabla_x \cdot (N_i U_i \otimes U_i) + \nabla_x (N_i T_n) + N_i \frac{\nabla_x \psi}{m_n} &= -\beta_i \frac{\nu_i}{\eta} N_i (U_i - U_n), \\ \frac{\partial}{\partial t} N_e + \nabla_x \cdot (N_e U_e) &= 0, \\ \beta_e^2 \frac{\partial}{\partial t} (N_e U_e) + \beta_e^2 \nabla_x \cdot (N_e U_e \otimes U_e) + \nabla_x (N_e T_n) - N_e \frac{\nabla_x \psi}{m_n} &= -\beta_e \frac{\nu_e}{\eta} N_e (U_e - U_n). \end{aligned}$$

Recall that here  $\psi$  is solution to the Poisson equation.

The Chapman–Enskog method consists in substituting the expansion  $U_\alpha^0 + \eta U_\alpha^1 + \eta^2 \dots$  truncated at first order in this system and insisting that this truncated

expansion be consistent to the highest possible order with the real solution. This results of course in a system of PDEs for the profiles  $N_\alpha$  and this system will be the desired approximation. Hence consider the truncated Chapman–Enskog expansion

$$U_\alpha \simeq U_\alpha^0 + \eta U_\alpha^1 \tag{3.22}$$

which is substituted to  $U_\alpha$  in the momentum equations for both the ions and the electrons. Balancing order by order in  $\eta$ , one finds

$$U_\alpha^0 = U_n \tag{3.23}$$

and, at order one for the ion velocity, taking account of (3.23)

$$U_\alpha^1 = -\frac{\beta_\alpha}{\nu_\alpha(T_n)N_\alpha} \left( N_\alpha \frac{\partial}{\partial t} U_n + N_\alpha U_n \cdot \nabla_x U_n + \frac{1}{\beta_\alpha^2} \nabla_x(N_\alpha T_n) + \frac{1}{\beta_\alpha^2} N_\alpha G \right). \tag{3.24}$$

This relation may be simplified by using the momentum conservation for the neutral particles, that is to say

$$\frac{\partial}{\partial t} U_n + U_n \cdot \nabla_x U_n = -\frac{1}{N_n} \nabla_x P_n \tag{3.25}$$

so we get, in the end:

$$U_\alpha^1 = -\frac{1}{\nu_\alpha(T_n)\beta_\alpha N_\alpha} \left( \nabla_x(N_\alpha T_n) + GN_\alpha - \beta_\alpha^2 \frac{N_\alpha}{N_n} \nabla_x P_n \right). \tag{3.26}$$

The system of equations for the approximate evolution of  $N_i$  and  $N_e$  is then obtained by substituting  $U_i^0 + \eta U_i^1$  and  $U_e^0 + \eta U_e^1$  to respectively  $U_i$  and  $U_e$  in the conservation of mass for the ions and the electrons.

With this prescription we see that the solution of the multifluid model ( $\mathcal{MF}$ ) may be approximated by the solution of the following diffusion model which reads as

$$\begin{aligned} \frac{\partial}{\partial t} N_i + \nabla_x \cdot (N_i U_n) &= \nabla_x \cdot \left[ \frac{\eta}{\nu_i \beta_i} \left( \nabla_x(N_i T_n) + N_i \frac{\nabla_x \psi}{m_n} - \beta_i^2 \frac{N_i}{N_n} \nabla_x(N_n T_n) \right) \right], \\ \frac{\partial}{\partial t} N_e + \nabla_x \cdot (N_e U_n) &= \nabla_x \cdot \left[ \frac{\eta}{\nu_e \beta_e} \left( \nabla_x(N_e T_n) - N_e \frac{\nabla_x \psi}{m_n} - \beta_e^2 \frac{N_e}{N_n} \nabla_x(N_n T_n) \right) \right]. \end{aligned}$$

This system of the form already presented in the multispecies diffusion model ( $\mathcal{D}$ ), but with the following choices of diffusion coefficients:

$$D_\alpha^T = 0, \quad D_\alpha = \eta \frac{T_n}{\nu_\alpha(T_n)}. \tag{3.27}$$

In this derivation of the multispecies diffusion model, there is no thermal diffusion. This is because we have started from multifluid model which is based on a Maxwellian closure for the charged particles. This is the main difference with the diffusion approximation obtained from the kinetic theory.

**Remark 3.1.** According to the expression of the ion velocity  $U_n + \eta U_i^1$  and the electron velocity  $U_n + \eta U_e^1$ , we can get an evaluation of the electric current  $J = N_i(U_n + \eta U_i^1) - N_e(U_n + \eta U_e^1)$  (by neglecting the  $\beta_e^2$  term)

$$J = (N_i - N_e)U_n + \frac{\eta}{\nu_e \beta_e N_e} \left( \nabla_x(N_e T_n) - N_e \frac{\nabla_x \psi}{m_n} \right) - \frac{\eta}{\nu_i \beta_i N_i} \left( \nabla_x(N_i T_n) + N_i \frac{\nabla_x \psi}{m_n} - \beta_i^2 \frac{N_i}{N_n} \nabla_x P_n \right),$$

where  $\psi$  solves the Poisson equation. Of course, the main term here is related to the electron part, that is to say  $J \simeq \frac{\eta}{\nu_e \beta_e N_e} (\nabla_x(N_e T_n) - N_e \nabla_x \psi / m_n)$ .

**Remark 3.2.** Approximating solutions of the multifluid model ( $\mathcal{MF}$ ) by solutions of the multispecies diffusion model ( $\mathcal{D}$ ) or ( $\mathcal{DF}$ ) makes sense only as long as one is dealing with smooth solutions. It is worth noticing that the characteristic speeds of the multispecies model are  $U_i \pm \sqrt{T_n}$  for the ions and  $U_e \pm \sqrt{T_n}$  for the electrons, while the drift in any of the two diffusion equations is  $U_n \simeq U_i \simeq U_e$  in the strong relaxation limit. Hence this theory cannot describe situations where the ion or electron densities have shocks or even high frequency oscillations.

#### 4. A Mathematical Justification of the Ambipolar Diffusion Equation

In this section, we shall attempt to give a mathematical justification of the ambipolar diffusion model ( $\mathcal{APD}$ ). Unfortunately, we have not been able to derive the ambipolar diffusion model completely from the kinetic model. We shall instead start from ( $\mathcal{DB}$ ). On the other hand, we can deal with a more general situation than in ( $\mathcal{DB}$ ): we assume that the ionic temperature  $T_n$  which is now denoted by  $T$  and the electronic temperature  $T_e$  are not equal but are only proportional, that is to say, there exists a positive constant  $r$  such that:

$$T_e = rT. \tag{4.1}$$

Let us recall that this model corresponds to the small electronic mass and strong ionic relaxation limit. We assume that there is no thermal diffusion, that is,  $D_i^T = 0$  in ( $\mathcal{DB}$ ). We shall introduce a further simplification: the time derivative is replaced by an implicit discretization where  $N_i^O$  is the ionic density at the beginning of the time-step and  $\lambda = 1/\Delta t$ . This is justified; indeed the time evolution of the ion density is very low compare to the coupling phenomena between electron and ion (which are instantaneous in this model).

As it was stated in Sec. 2, after introducing the Debye length  $l_D$ , the Poisson equation reads as

$$-\frac{N_C}{T_C} l_D^2 \Delta \psi = N_i - N_e.$$

We assume here that the Debye length is very small compared to the characteristic length  $l_M$ :

$$\epsilon = l_D/l_M \ll 1.$$

We use a dimensionless spatial variable related to  $l_M$ . We also normalize the density function  $N_i$  and  $N_e$  by the characteristic value  $N_C$ , the potential  $\psi$  by the characteristic temperature  $\mathcal{T}_C$  and the function  $T$  by the characteristic value  $m_n \mathcal{T}_C$ . Recall that

$$P_i = N_i T, \quad P_i^O = N_i^O T, \quad P_e = r N_e T.$$

The resulting equations are recast in dimensionless form

$$\lambda N_i - \nabla \cdot \left[ D_i \left( \nabla P_i + \frac{P_i}{T} \nabla \psi \right) \right] = \lambda N_i^O, \tag{4.2}$$

$$\nabla P_e = \frac{P_e}{rT} \nabla \psi, \tag{4.3}$$

$$-\epsilon^2 \Delta \psi = N_i - N_e. \tag{4.4}$$

In the above system, the temperature  $T$  and the diffusion coefficient  $D_i$  are given positive functions of  $x$ . This section is aimed at giving a mathematical derivation of the ambipolar diffusion equation starting from this system. Although, as we have already said, this system is not itself a first principle, we think that it is worthwhile to present rigorous arguments in favor of the ambipolar diffusion model, in particular to clarify the limits of its validity. Eliminating the electric field, thanks to (4.3), leads to the following coupled system where  $P_i = P_i^\epsilon$  and  $P_e = P_e^\epsilon$  are the new unknowns:

$$\frac{\lambda}{T} P_i^\epsilon - \nabla_x \cdot [D_i (\nabla_x P_i^\epsilon + r P_i^\epsilon \nabla_x (\log P_e^\epsilon))] = \frac{\lambda}{T} P_i^O. \tag{4.5}$$

$$-\epsilon^2 \nabla_x \cdot (T \nabla_x (\log P_e^\epsilon)) = \frac{1}{rT} \left( P_i^\epsilon - \frac{1}{r} P_e^\epsilon \right). \tag{4.6}$$

The above system, which is the classical nonlinear Poisson equation, is posed in a domain  $O \subset \mathbb{R}^3$  with smooth boundary  $\partial O$ . The first reduced system (4.2)–(4.4) is supplemented by the following homogeneous Neumann conditions at the boundary  $(\frac{\partial \psi}{\partial n})|_{\partial O} = 0$  (indicating that the boundary is insulating) and  $(\frac{\partial N_e}{\partial n})|_{\partial O} = 0$  (indicating that particles are reflected at the boundary). These conditions, with the relation (4.3) show that  $(\frac{\partial T}{\partial n})|_{\partial O} = 0$ . Then, it leads to the following boundary conditions for the reduced system (4.5), (4.6)

$$\left( \frac{\partial P_i}{\partial n} \right) \Big|_{\partial O} = \left( \frac{\partial P_e}{\partial n} \right) \Big|_{\partial O} = 0. \tag{4.7}$$

One might say that these boundary conditions are not extremely interesting from a physical viewpoint. However, (as we shall see below) the ambipolar diffusion model is a consequence of quasi-neutrality. In order to establish the validity of the ambipolar diffusion model up to the boundary of the domain  $O$  it is important that

the boundary conditions exclude the possibility of source terms at the boundary which would induce a local breakdown of the quasi-neutrality condition. The general mathematical analysis involving boundary layers describing the departure from quasi-neutrality near the boundary is a rather difficult subject. We refer to Ref. 5, for the particular case where the temperature is constant and the ionic density prescribed. For a boundary layer analysis of quasi-neutrality breakdown in the weakly ionized case, see Ref. 1 and the work Ref. 16. In the sequel, we shall study the system (4.5)–(4.7) in the limit as  $\epsilon \rightarrow 0$ .

**4.1. Maximum principle**

What is fundamental for our purpose is to establish bounds on the electronic and ionic pressures that are uniform as  $\epsilon \rightarrow 0$ . The special structure of the system (4.5)–(4.7) and particularly assumption (4.1) allow a change of unknowns analogous to the introduction of Slotboom variables in the diffusion equations for semiconductors (see for example Ref. 17).

**Proposition 4.1.** *Assume that the data  $P_i^O$  and  $T$  in (4.5)–(4.7) verify the bounds*

$$\frac{1}{C_0} \leq P_i^O(x) \leq C_0, \quad T(x) > 0, \quad x \in O, \tag{4.8}$$

for some positive constant  $C_0$ . Then, there exists a positive constant  $C_1$  such that, for all  $\epsilon > 0$ , any weak solution  $P_i^\epsilon > 0, P_e^\epsilon > 0$  of the system (4.5)–(4.7) satisfies the bounds

$$\frac{1}{C_1} \leq P_i^\epsilon(x), \quad P_e^\epsilon(x) \leq C_1, \quad x \in O. \tag{4.9}$$

**Proof.** Let  $P_i > 0, P_e > 0$  be a weak solution of the system (4.5)–(4.7). We define the analogue of a Slotboom variable, that is, a function  $u$  such that

$$P_i = uP_e^{-r}.$$

Equation (4.5) then becomes

$$\frac{\lambda}{T} u P_e^{-r} - \nabla_x \cdot (D_i P_e^{-r} \nabla_x u) = \frac{\lambda}{T} P_i^O. \tag{4.10}$$

The classical maximum principle, when applied to Eq. (4.10), gives the estimate

$$\sup_{x \in O} u(x) \leq \sup_{x \in O} P_i^O(x) \cdot \left( \sup_{x \in O} P_e(x) \right)^r.$$

Applying next the same maximum principle to (4.6) gives

$$\frac{1}{r} \left( \sup_{x \in O} P_e(x) \right)^{r+1} \leq \sup_{x \in O} u(x).$$

Hence one has the bounds

$$\frac{1}{r} \sup_{x \in O} P_e(x) \leq \sup_{x \in O} P_i^O(x)$$

and

$$\sup_{x \in O} u(x) \leq \left( \sup_{x \in O} P_i^O(x) \right)^{r+1}.$$

Notice that these bounds are uniform in  $\epsilon$ . Proceeding exactly in the same manner with the minimum principle gives lower bounds on  $P_e$  and  $u$  in terms of  $\inf_{x \in O} P_i^O(x)$ . □

**Remark 4.1.** The uniform estimate above and classical arguments allow to prove the existence for all  $\epsilon > 0$  of a weak solution of the system (4.5)–(4.7) satisfying (4.10) under the assumption (4.8); we shall not dwell on this and take this existence result for granted in the sequel.

**4.2. The asymptotic limit as  $\epsilon \rightarrow 0$**

In this subsection, we first show a result related to quasi-neutrality, then the desired result on the ambipolar diffusion.

**Proposition 4.2.** *Assume that the data  $P_i^O$  and  $T$  in (4.5)–(4.7) verify the bounds (4.8) for some positive constant  $C_0$ . Then the solution  $P_i^\epsilon, P_e^\epsilon$  of (4.5)–(4.7) and the corresponding ionic and electronic densities  $N_i^\epsilon$  and  $N_e^\epsilon$  satisfy the asymptotic estimate*

$$\|P_i^\epsilon - rP_e^\epsilon\|_{L^2} = O(\epsilon), \quad \|N_i^\epsilon - N_e^\epsilon\|_{L^2} = O(\epsilon), \tag{4.11}$$

as  $\epsilon \rightarrow 0$ .

**Proof.** Multiplying (4.6) by  $\log(P_i^\epsilon) - \log(\frac{1}{r}P_e^\epsilon)$  and integrating with respect to  $x$  results in:

$$\begin{aligned} & \int_O \frac{1}{rT} \left( P_i^\epsilon - \frac{1}{r}P_e^\epsilon \right) \left( \log(P_i^\epsilon) - \log\left(\frac{1}{r}P_e^\epsilon\right) \right) dx \\ & \leq \epsilon^2 \int_O T \nabla_x \log P_e^\epsilon \cdot (\nabla_x \log P_i^\epsilon - \nabla_x \log P_e^\epsilon) dx. \end{aligned} \tag{4.12}$$

Multiplying then (4.5) by  $(\log P_i^\epsilon + r \log P_e^\epsilon)$  and integrating with respect to  $x$  gives

$$\begin{aligned} & \int_O D_i P_i^\epsilon |\nabla_x \log P_i^\epsilon + r \nabla_x \log P_e^\epsilon|^2 dx \\ & \leq \int_O \frac{\lambda(P_i^O - P_i^\epsilon)}{T} (\log P_i^\epsilon + r \log P_e^\epsilon) dx. \end{aligned} \tag{4.13}$$

It follows from this inequality and the uniform estimates (4.1) that

$$\|\nabla_x \log P_i^\epsilon + r \nabla_x \log P_e^\epsilon\|_{L^2} \leq C_2 \tag{4.14}$$

uniformly in  $\epsilon$ . Equation (4.12) is then transformed into

$$\begin{aligned} & \int_O \frac{1}{rT} \left( P_i^\epsilon - \frac{1}{r} P_e^\epsilon \right) \left( \log(P_i^\epsilon) - \log\left(\frac{1}{r} P_e^\epsilon\right) \right) dx + (r+1)\epsilon^2 \int_O T |\nabla_x \log P_e^\epsilon|^2 dx \\ & \leq \epsilon^2 \int_O T \nabla_x \log P_e^\epsilon \cdot (\nabla_x \log P_i^\epsilon + r \nabla_x \log P_e^\epsilon) dx. \end{aligned} \tag{4.15}$$

The first integral appearing on the left-hand side of the inequality above is non-negative; therefore, by using the Cauchy–Schwarz inequality with (4.14), one deduces from (4.15) the existence of a positive constant  $C_3$  (uniform in  $\epsilon$ ) such that

$$\|\sqrt{T} \nabla_x \log P_e^\epsilon\|_{L^2} \leq C_3. \tag{4.16}$$

Injecting the bound (4.16) in (4.15) gives the existence of a positive constant  $C_4$  such that

$$\int_O \frac{1}{rT} \left( P_i^\epsilon - \frac{1}{r} P_e^\epsilon \right) \left( \log(P_i^\epsilon) - \log\left(\frac{1}{r} P_e^\epsilon\right) \right) dx \leq C_4 \epsilon^2.$$

It is classical to deduce from this inequality that

$$\left\| \sqrt{P_i^\epsilon} - \sqrt{\frac{1}{r} P_e^\epsilon} \right\|_{L^2} = \|\sqrt{T}(\sqrt{N_i^\epsilon} - \sqrt{N_e^\epsilon})\|_{L^2} = O(\epsilon);$$

with the uniform bounds (4.9) established in Proposition 4.1, this gives precisely the announced estimate. □

**Theorem 4.1.** *Assume that the data  $P_i^O$  and  $T$  verify the bounds (4.8) for some positive constant  $C_0$  and that  $T \in W^{1,\infty}(O)$ . For all  $\epsilon > 0$  let  $N_i^\epsilon$  and  $N_e^\epsilon$  be respectively the ion and electron densities corresponding to the solution  $P_e^\epsilon, P_i^\epsilon$  of the system (4.5)–(4.7) verify*

$$N_i^\epsilon \rightarrow N_*, \quad N_e^\epsilon \rightarrow N_*$$

*in  $L^2(O)$  as  $\epsilon \rightarrow 0$ . The function  $N_*$  is a solution of the “ambipolar diffusion equation”*

$$\lambda N_* - \nabla_x \cdot (D_i \nabla_x (N_* (T_n + T_e))) = \lambda N_i^O \tag{4.17}$$

*on the domain  $O$ , with the Neumann boundary condition*

$$\left. \frac{\partial N_*}{\partial n} \right|_{\partial O} = 0. \tag{4.18}$$

**Proof.** We go back to (4.5) which is conveniently recast in the form

$$\lambda N_i^\epsilon - \nabla_x \cdot \left[ \frac{D_i}{m_i} \left( \nabla_x (N_i^\epsilon T_n) + \frac{N_i^\epsilon}{N_e^\epsilon} \nabla_x (N_e^\epsilon T_e) \right) \right] = \lambda N_i^O.$$

Letting  $\epsilon \rightarrow 0$  and appealing to the estimate (4.9), one easily obtains further uniform estimates as follows:

$$\|\nabla_x(N_i^\epsilon T_n)\|_{L^2}, \|\nabla_x(N_e^\epsilon T_e)\|_{L^2} \leq C_5, \tag{4.19}$$

where  $C_5$  is some positive constant independent of  $\epsilon$ . Estimates (4.11) and (4.9) give

$$\left\| \frac{N_i^\epsilon}{N_e^\epsilon} - 1 \right\|_{L^2} = O(\epsilon) \tag{4.20}$$

as  $\epsilon \rightarrow 0$ . It follows from (4.19) and (4.20) that

$$\left\| \nabla_x(N_i^\epsilon T_n) + \frac{N_i^\epsilon}{N_e^\epsilon} \nabla_x(N_e^\epsilon T_e) - \nabla_x(N_i^\epsilon T_n) - \nabla_x(N_e^\epsilon T_e) \right\|_{L^2} = O(\epsilon).$$

Hence, as  $\epsilon \rightarrow 0$ , the ionic and electronic densities  $N_i^\epsilon$  and  $N_e^\epsilon$  converge weakly in  $L^2(O)$  to the same function  $N_*$  satisfying Eq. (4.17) with the Neuman boundary condition (4.18). This weak convergence follows from the uniform bounds in  $\epsilon$  established in Proposition 4.1 and from the uniqueness of the solution of the ambipolar diffusion problem (4.17) and (4.18). This weak convergence is in fact strong because of the estimates (4.19) and by the regularity assumed on  $T$ . □

### Appendix A. Variational Formulation for the Diffusion Coefficients

Let us first recall an elementary property of the mass diffusion coefficient  $\tilde{D}_\alpha$ .

**Lemma A.1.** *The diffusion coefficient of each species is positive:  $\tilde{D}_\alpha > 0$ .*

**Proof.** Indeed, by Proposition 3.1(b)

$$\tilde{D}_\alpha = \frac{T_n}{3N_n} \langle c_\alpha(|W|)|W|^2|1 \rangle = -\frac{T_n}{3N_n} \sum_{i=1}^3 \langle c_\alpha(|W|)W_i \Gamma[c_\alpha(|W|)W_i] \rangle \geq 0.$$

If  $\tilde{D}_\alpha = 0$ , again by Proposition 3.1,  $c(|W|)W_i \in \text{Ker } \Gamma$  for  $i = 1, 2, 3$ ; but this is impossible since by definition  $c(|W|)W_i \in (\text{Ker } \Gamma)^\perp$  : see Lemma 3.2. □

This Appendix is aimed at giving a variational formulation of the integral equations in Lemma 3.2 which is useful in numerical computations of the diffusion coefficients. The problem that we have to solve can be stated as follows: Find a radial function  $X$  which satisfies the integral equations

$$-\Gamma(X(|W|)W_i) = h(|W|^2)W_i, \quad i = 1, 2, 3 \tag{A.1}$$

in the following two cases:

- case 1:  $h \equiv 1$ ; then  $X = c_\alpha$  and  $\tilde{D}_\alpha = \frac{T_n}{3N_n} \langle X(|W|)|W|^2 \rangle$ .
- case 2:  $h \equiv |W|^2 - 5$ ; then  $X = a_\alpha(|W|)(|W|^2 - 5)$  and

$$\tilde{D}_\alpha^T = \frac{T_n}{6N_n} \langle X(|W|)|W|^2 \rangle.$$

That the problem is well-posed, follows from  $\langle h(|W|^2)W_i|1 \rangle = 0$ .

**Lemma A.2.** *The variational formulation equivalent to (A.1) is*

$$\inf_{\theta \in (\text{Ker } \Gamma)^\perp} -\frac{1}{2} \langle \Gamma(\theta)W_i | \theta \rangle - \langle \theta | h(|W|^2)W_i \rangle, \quad i = 1, 2, 3. \tag{A.2}$$

In other words,  $X$  is the only radial function such that

$$\forall \theta \in (\text{Ker } \Gamma)^\perp, \quad -\langle \Gamma(X(|W|)W_i) | \theta \rangle = \langle h(|W|^2)W_i | \theta \rangle, \quad i = 1, 2, 3. \tag{A.3}$$

**Proof.** Observe that  $-\Gamma$  is Fredholm self-adjoint non-negative and hence induces an elliptic bilinear form on  $(\text{Ker } \Gamma)^\perp$ . The result follows from the Lax–Milgram theorem.  $\square$

The variational problem (A.2) is then approximated by the Galerkin method. It has been known for a long time (see for example Ref. 7) that the Sonine–Laguerre polynomials provide a good Galerkin basis for this problem; these polynomials are<sup>a</sup>

$$S^{(k)}(t) = \sum_{p=0}^k \frac{(3/2 + k)!}{(3/2 + p)!p!(k - p)!} 2^{-p/2} (-t)^p.$$

Then, if  $H_n$  denotes the subspace spanned by the functions  $S^0(|W|^2)$ ,  $S^1(|W|^2), \dots, S^n(|W|^2)$ , the  $n$ th order approximation  $\hat{X}$  of the solution  $X$  of this problem satisfies

$$\forall \theta \in H_n, \quad -\langle \Gamma(\hat{X}(|W|)W_i) | \theta \rangle = \langle h(|W|^2)W_i | \theta \rangle, \quad i = 1, 2, 3. \tag{A.4}$$

Recall the orthogonal property of these polynomials

$$\int_0^\infty t^{3/2} e^{-t} S^{(m)}(\sqrt{2}t) S^{(k)}(\sqrt{2}t) dt = \delta_{mk} \frac{(3/2 + k)!}{k!}.$$

It is translated as follows for the scalar product  $\langle \cdot | \cdot \rangle$ :

$$\langle S^{(m)}(|W|^2)W_i | S^{(k)}(|W|^2)W_j \rangle = \delta_{ij} \delta_{mk} \frac{1}{3\sqrt{\pi}} \frac{(3/2 + k)!}{k!}.$$

Return to the variational problem (A.2) for a fixed  $i$ ; its  $n$ th order approximation (A.4) has the following solution

$$\hat{X}(W) = \sum_{k=0}^{n-1} \xi_k S^{(k)}(|W|^2),$$

where the vector  $\xi = (\xi_0, \dots, \xi_{n-1})$  is the solution of the linear system

$$-\sum_{k=0}^{n-1} \langle \Gamma[S^{(k)}(|W|^2)W_i] | S^{(m)}(|W|^2)W_i \rangle \xi_k = L_m \tag{A.5}$$

with

$$L_m = \langle h(|W|^2)W_i | S^{(m)}(|W|^2)W_i \rangle, \quad m = 0, \dots, n - 1.$$

<sup>a</sup>For any  $x$ , real,  $x!$  is defined in term of Gamma function, e.g.  $(3/2)! = 3\sqrt{\pi}$ .

Notice that  $\langle \hat{X}(W) || W|^2 \rangle = \xi_0$  (since  $\langle S^{(k)} || W|^2 \rangle = 0$  for any  $k \neq 0$ ). Then specializing this to the case where  $h = 1$  and the case where  $h(|W|) = |W|^2 - 5$  readily gives the following approximations of the diffusion coefficients at order  $n$

$$\tilde{D}_\alpha \simeq \frac{T_n}{3N_n} \xi_{0,c}, \quad \tilde{D}_\alpha^T \simeq \frac{T_n}{6N_n} \xi_{0,a},$$

where  $\xi_{0,c}$  is the first component of the solution  $\xi$  of (A.5) with  $L = (-\frac{1}{2}, 0, 0, \dots)$  and  $\xi_{0,a}$  is the first component of the solution  $\xi$  of (A.5) with  $L = (0, \frac{5}{4}, 0, \dots)$ . We refer to Ref. 16 for detailed computations.

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